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# Twisting procedure on torogonal structures

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The Cliffordian formalism gives a natural simple approach of spin geometry through torogonal structures. The pure spinor theory enables one to construct explicitly a torogonal lifting of a certain reduction of a pseudo-Riemannian structure. The twisting procedure intervenes in different useful schemes in geometric quantization when the usual Kostant–Souriau procedure breaks down.

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## 1. Introduction

The crucial role played by spin geometry in Riemannian geometry proceeds from the interaction between theoretical physics and geometry. B. Riemann disengaged the concept of manifold endowed with a general length element and showed that a 4-covariant tensor, the so-called curvature tensor, is one of its fundamental invariants. Afterwards, Riemannian geometry appeared as a theory with a particular wealth for various reasons. Some of them are connected to the development of one of its variants, namely the Lorentzian geometry, which is the mathematical framework of general relativity. But other ones are purely mathematical due to the generality and the power of the natural differential tools that it develops, such as the Levi-Civita covariant derivation. On the other hand, from a strictly geometrical point of view, this theory contains natural examples which may be able to supply models. Thus in field theory, an important criterion for a Riemannian manifold to be a reasonable model of spacetime is that it admits spinors. These objects were introduced via a well known matricial approach by Pauli to modelize the internal angular moment of the electron, the so-called spin, and by Dirac to linearize the Klein–Gordon operator in quantum mechanics. E. Cartan's works on the group representations and above all Chevalley's book brought the mathematical tools which permitted a better understanding of their


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meaning as elements of representation spaces for Clifford algebras.

In section 2, the consistent use of the Cliffordian formalism which reflects the inner symmetries and basic identities of the Riemannian structure, gives a natural simple approach of spin geometry and enables one to abandon physically superfluous restrictions through what I proposed to call the torogonal structures, which extend the well known spin$^c$ structures [1,13]. The generalization consists in assuming that the underlying manifold is not necessarily orientable and its metric tensor has a general pseudo-Riemannian signature.

Recall that the fundamental invariant of a vector space with a quadratic form is its associated Clifford algebra. The globalization of this concept on a pseudo-Riemannian manifold is always realized through the Clifford bundle notion. On the other hand, the globalization of the spinor algebraic concept, for which spinors are elements of a complex vector space of any irreducible representation of Clifford algebras, leads naturally to the S.$\mathbb{C}$. structures notion introduced by G. Karrer [10], but requires topological restrictions on the manifold. But there exists another way for this globalization using the lifting of structure groups [6,7]. For this, we seek to construct liftings of the orthonormal frame bundle in relation to either a double covering (the pinorial group) or a central extension by the 1-torus $S^1$ of the orthogonal group, from where the utilized terminology, namely the torogonal group, in order to obtain some important things, such as pinorial or torogonal structures and connections. These groups which appear in a certain torogonal diagram are not the conventional ones since they are explicitly constructed from the complexification of the Cliffordian formalism and the corresponding pinorial (resp. torogonal) obstruction to such globalization is simpler than the conventional one and it is measured by a degree two (resp. three) cohomology class in the second (resp. third) Čech cohomology group $H^2(M, \mathbb{Z}_2)$ (resp. $H^3(M, \mathbb{Z})$) with $\mathbb{Z}_2$ (resp. $\mathbb{Z}$) coefficients. Torogonal and S.$\mathbb{C}$. structures are equivalent, and the spinors we construct all come from these, as sections of associated vector bundles.

Our purpose in section 3 is to draw attention to circumstances under which torogonal structures arise naturally in the presence of a certain distinguished reduction of the orthonormal frame bundle or equivalently, of some maximal isotropic subbundle of the complexified tangent bundle. In this context, we shall mainly use the pure spinor concept of Chevalley–Cartan [2,3]. Since the Clifford product of elements of any basis of some complex maximal isotropic vector subspace generates a minimal left or right ideal, we obtain a pure spinor taking the intersection of such objects.

In section 4, since the cohomology group $H^1(M, S^1)$ classifies the set of inequivalent torogonal structures, according to the complex line bundle theory mainly developed by B. Kostant and J.M. Souriau [12,15], we can twist the previous torogonal lifting by an adequate integral cohomology class in order to find the different schemes utilized in geometric quantization.

In section 5, we have assembled some useful examples. The previous construction is available on compatible almost complex structure and it is canonical on symplectic manifold with respect to any subordinate almost Hermitian structure. We explain why using twisting procedure on torogonal structures resolves all problems which occur in the quantization of energy surfaces of the harmonic oscillator. Finally, using the techniques of Yano–Ishihara [17], we give some examples taken from mechanical systems.

## 2. Torogonal structures

Let $M$ be a pseudo-Riemannian manifold endowed with a nonsingular metric $Q$ of any signature. We shall model $(M, Q)$ on a real vector space $\mathbb{E}$ equipped with an inner product of same signature and denoted by the same letter $Q$. The bundle $O(M, Q)$ of orthonormal frames or simply $O(M)$ then has structure group $O(Q)$: the group of orthogonal automorphisms of $(\mathbb{E}, Q)$.

### 2.1. THE TOROGONAL DIAGRAM

We can form the Clifford algebra $C(Q)$ of the pseudo-Euclidean vector space $(\mathbb{E}, Q)$ as the quotient of the tensor algebra of $\mathbb{E}$ by the two-sided ideal generated by elements of the form $x\otimes x+Q(x, x)\cdot 1$ where $x\in\mathbb{E}$. Its complexification $C(Q)\otimes_{\mathbb{R}}\mathbb{C}$ is isomorphic to the Clifford algebra $C(Q')$ constructed from the complexification $(\mathbb{E}', Q')$. Let $C^*(Q)$ (resp. $C^*(Q')$) denote the group of invertible elements in $C(Q)$ (resp. $C(Q')$).

The complex twisted Clifford group $\Gamma(Q')$ is the subgroup of $C^*(Q')$ consisting of the elements $g$ which satisfy $\alpha(g)xg^{-1}\in\mathbb{E}'$ for all $x\in\mathbb{E}'$, where $\alpha$ is the degree involution induced by $-\mathrm{id}_{\mathbb{E}'}$. We have the following exact sequence:

$$1\to\mathbb{C}^*\to\Gamma(Q')\xrightarrow{p} O(Q')\to 1\ ,$$

where $p$ is the restriction to $\Gamma(Q')$ of the corresponding twisted adjoint representation of $C^*(Q')$ in $C(Q')$, and $O(Q')$ is the complex orthogonal group.

The homomorphism $N$: $\Gamma(Q')\to\mathbb{C}^*$ defined by $N(g)=\beta(g)\,g$, where $\beta$ is the anti-automorphism induced by the natural embedding of $\mathbb{E}'$ in the opposite Clifford algebra, is the spinor norm morphism.

Taking the following adequate inverse images we obtain the pinorial group $\mathrm{pin}(Q)=\ker N\cap p^{-1}(O(Q))$ and the torogonal group $\Delta(Q)=N^{-1}(S^1)\cap p^{-1}(O(Q))$ which are respectively a double covering of $O(Q)$ and its central extension by the one-dimensional torus $S^1$; thus, we get a commutative diagram of Lie groups with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & S^0=\{\pm 1\} & \longrightarrow & \mathrm{pin}(Q) & \xrightarrow{\sigma} & O(Q) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & S^1 & \longrightarrow & \Delta(Q) & \xrightarrow{\delta} & O(Q) & \longrightarrow & 1 \\
 & & \cdot^2\downarrow & & N\downarrow & & & & \\
 & & S^1 & = & S^1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
$$

where $\sigma$ and $\delta$ denote the restriction of $p$ to $\mathrm{pin}(Q)$ and $\Delta(Q)$, and $N$ restricted to $S^1$ is the squaring map $\cdot^2$: $S^1 \to S^1/S^0 = S^1$.

Another interpretation of the torogonal diagram consists of identifying the torogonal group with the quotient $(\mathrm{pin}(Q)\times S^1)/S^0$ of the product $\mathrm{pin}(Q)\times S^1$ with respect to the diagonal subgroup $S^0$. The homomorphism $\delta$ (resp. $N$) is induced by the map which sends $(g, z)$ to $\sigma(g)$ (resp. $z^2$) for $g\in \mathrm{pin}(Q)$ and $z\in S^1$.

Taking the composite of $\beta$ with the complex conjugation we obtain, from the unitary subgroup $U$ of $C^*(Q')$ characterized by $\beta(\bar{g})g=1$, the unitary torogonal (resp. pinorial) group $\Delta^U(Q)=\Delta(Q)\cap U$ (resp. $\mathrm{pin}^U(Q)=\mathrm{pin}(Q)\cap U$). With respect to the Cartan–Dieudonné decomposition of isometries belonging to the direct image $O^U(Q)=\sigma(\mathrm{pin}^U(Q))=\delta(\Delta^U(Q))$, there is an even number of real negative hyperplane symmetries. If we specialize to the model of spacetime where $M$ is a four-dimensional manifold endowed with a metric having signature (3, 1), then $O^U(3, 1)$ is nothing but that of the Lorentz orthochronal group.

**Remarks.** These constructions are available for even or odd dimensional vector spaces. The conventional pinorial groups can be obtained from the real twisted Clifford groups.

### 2.2. TOROGONAL AND PINORIAL STRUCTURES

**Definitions.** A torogonal (resp. pinorial) structure for $(M, Q)$ is a principal $\Delta(Q)$ (resp. pin $(Q)$) bundle $\Delta(M)$ (resp. pin $(M)$) together with a map $\delta$: $\Delta(M)\to O(M)$ (resp. $\sigma$: pin $(M)\to O(M)$) of principal bundles equivariant with respect to the torogonal projection $\delta$ (resp. the covering map $\sigma$). Two such structures are equivalent if there is a strong bundle isomorphism between them which intertwines the corresponding bundle maps.

**Torogonal and pinorial obstructions.** We will use Čech cohomology with coefficients in sheaves of not necessarily Abelian groups [5]. Recall that isomorphism classes of principal bundles such that $[O(M)]$ are represented by elements in the sheaf cohomology group $H^1(M, O(Q))$ induced for example by a system of transition functions of $O(M)$. From the well known cohomology long exact sequences induced by the previous exact sequences of groups, we can define the torogonal (resp. pinorial) Bockstein $B_\delta^1: H^1(M, O(Q)) \to H^2(M, S^1)$ (resp. $B_\sigma^1$: $H^1(M, O(Q)) \to H^2(M, S^0)$) whence the torogonal (resp. pinorial) obstruction $B_\delta^1\,[O(M)]$ (resp. $B_\sigma^1\,[O(M)]$).

From the cohomology sequence induced by the exact sequence of groups

$$0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{m} \mathbb{C}^* \to 1$$

where $m(z) = \exp(2i\pi z)$, and due to the flatness of the $\mathbb{C}$-coefficients sheaf, we have the following Bockstein isomorphisms $B_m^1$: $H^1(M, \mathbb{C}^*) \xrightarrow{\sim} H^2(M, \mathbb{Z})$ and $B_m^2$: $H^2(M, \mathbb{C}^*) \xrightarrow{\sim} H^3(M, \mathbb{Z})$.

Henceforth, identifying $S^0$ with $\mathbb{Z}_2$, we shall consider the following invariants, namely the torogonal (resp. pinorial) obstruction $\mathscr{W}_3 = B_m^2 \circ B_\delta^1\,[O(M)]$ (resp. $\mathscr{W}_2 = B_\sigma^1\,[O(M)]$) in $H^3(M, \mathbb{Z})$ (resp. $H^2(M, \mathbb{Z}_2)$).

**Theorem.** *There exists a torogonal (resp. pinorial) structure if and only if the torogonal (resp. pinorial) obstruction vanishes. Inequivalent torogonal (resp. pinorial) structures are in one to one correspondence with elements of* $H^1(M, S^1) \xrightarrow{\sim} H^2(M, \mathbb{Z})$ *(resp.* $H^1(M, S^0) \xrightarrow{\sim} H^1(M, \mathbb{Z}_2)$*).*

**Proposition.** *The pinorial obstruction is expressed directly in terms of Stiefel–Whitney classes* $W_i^\pm$ *of any Sylvester decomposition of the tangent bundle* $\mathrm{T}M = \mathrm{T}M^+ \oplus \mathrm{T}M^-$ *by the formula* $\mathscr{W}_2 = W_2^+ + W_2^-$.

*Proof.* We assume the signature of $Q$ is $(r, s)$ ($r$ positive and $s$ negative squares). With transparent notations and since the product $O(r) \times O(s)$ is a retract of $O(r, s)$ which lifts according to a homotopy equivalence of the twisted product $(\mathrm{pin}(r) \times \mathrm{pin}(s))/S^0$ with $\mathrm{pin}(r, s)$, the pinorial obstruction is given in terms of universal Stiefel–Whitney classes by

$$\mathscr{W}_2 = aW_2^+ + bW_2^- + c(W_1^+)^2 + dW_1^+ W_1^- + f(W_1^-)^2,$$

where $a$, $b$, $c$, $d$, $f$ are constant coefficients in $\mathbb{Z}_2$ independent of the metric and rank [9].

Suppose that $Q$ is positive definite ($s=0$); in the oriented case ($W_1^+ = 0$), we must have $a=1$; in the one-dimensional case, since the pinorial group which coincides with the conventional pinorial group is equal to $S^0 \oplus S^0$, any principal $O(1)$ bundle always admits a pinorial structure, whence $c=0$.

If $Q$ is negative definite ($r=0$), in the oriented case ($W_1^- = 0$) we still obtain $b=1$. In the one-dimensional case $\mathrm{pin}(0, 1) = S^0 \oplus S^0$, whence $f=0$; while, in contrast, the conventional pinorial group is equal to $\mathbb{Z}_4$, such that the corresponding Bockstein associated to the exact sequence

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$$

is nothing more than the square of the cup-product, whence here $f = 1$.

In order to determine the coefficient $d$, recall that for any given real vector bundle $\mathbb{F}$ equipped with a positive definite metric $q$, the Whitney sum $\mathbb{F} \oplus \mathbb{F}$ endowed with a neutral metric $q \oplus (-q)$ always admits conventional or nonconventional pinorial structure [16b]. Applying this property when $r = s = 1$, we find $d = 0$ in the nonconventional case, while in the conventional case we must have $d = 1$.

**Remark.** The nonconventional pinorial obstruction is simpler than the conventional one which is equal to $W_2^+ + W_2^- + W_1^+ W_1^- + (W_1^{-2})$.

### 2.3. CLIFFORD BUNDLES AND CLIFFORDIAN STRUCTURES

The real Clifford bundle $C(M)$ of $(M, Q)$ is the real algebraic bundle associated to the orthonormal frame bundle $O(M)$ under the group morphism ext: $O(Q) \to \mathrm{Aut}_{\mathbb{R}}(C(Q))$, where $\mathrm{ext}(u)$ is the natural extension of the isometry $u \in O(Q)$ to automorphism of the standard Clifford algebra $C(Q)$, by means of the universal property of Clifford algebras. We shall consider its complexified whose fibre at $x \in M$ is the complexified Clifford algebra $C_x(M) \otimes_{\mathbb{R}} \mathbb{C}$.

**Definition.** A Cliffordian structure or so-called S.ℂ. structure [10] consists of some complex vector bundle $\mathcal{S}$ over $M$ together with an algebraic bundle epimorphism h of the complexified Clifford bundle to the endomorphism bundle of $\mathcal{S}$; $\mathcal{S}$ is called the spinor bundle.

**Cliffordian obstruction.** We shall consider only even dimensional pseudo-Riemannian manifold so that h is an isomorphism and the standard Clifford algebra $C(Q')$ is a central simple algebra. According to the algebraic definition of spinors, the complex vector space $S$ of any irreducible representation $\rho$ of $C(Q')$ to $S$ can be chosen as standard spinors space. Since any automorphism of $C(Q')$ is an inner automorphism, we can obtain the exact sequence of Skolem–Noether

$$1 \to \mathbb{C}^* \to \mathrm{GL}_{\mathbb{C}}(S) \xrightarrow{f} \mathrm{Aut}_{\mathbb{C}}(C(Q')) \to 1 ,$$

where $f(u) = \rho^{-1} \circ I(u) \circ \rho$ for all $u \in \mathrm{GL}_{\mathbb{C}}(S)$ (the complex linear group of $S$) and $I(u)\ (v) = uvu^{-1}$ for all $v \in \mathrm{gl}_{\mathbb{C}}(S)$. If we take the composite of the Cliffordian Bockstein $B_f^1$ with the cohomological prolongation of the previous group morphism ext, we define the Cliffordian obstruction

$$B_m^2 \circ B_f^1 \circ \mathrm{ext}_* [O(M)] \quad \text{in } H^3(M, \mathbb{Z}) .$$

**Theorem**. *Cliffordian and torogonal obstructions coincide so that S.$\mathbb{C}$. structures and torogonal structures are equivalent.*

*Proof.* According to the Skolem–Noether theorem, any Cliffordian rotation can be interpreted by means of spinorial rotation or vectorial rotation, so that $f\circ p = \mathrm{ext}\circ\varphi$, where $\varphi$ is the nontwisted adjoint representation of $C^*(Q')$ in $C(Q')$ defined by $\varphi(g)(t) = gtg^{-1}$ [16b].

**Theorem 1**. *There exists a torogonal structure if and only if the pinorial obstruction is the mod 2 reduction of an integral class.*

*Proof.* Suppose that $\mathcal{W}_3 = 0$. Applying the spinor norm morphism to the transition functions of the torogonal bundle $\Delta(M)$, we obtain the so-called torogonal complex line bundle $N(M)$ as a principal $S^1$ bundle. Its isomorphism class is represented by an integral cohomology class $C_1 = B^1_m[N(M)]$ in $H^2(M, \mathbb{Z})$. If $\{U_\alpha\}$ is an open simple covering of $M$, let $\Delta_{\alpha\beta}$: $U_\alpha \cap U_\beta \to \Delta(Q)$ be a system of transition functions of $\Delta(M)$. On the contractible $U_\alpha \cap U_\beta$ it is always possible to choose a well defined square root $z_{\alpha\beta} = m(\theta_{\alpha\beta})$ of the transition functions $N\circ\Delta_{\alpha\beta}$ of $N(M)$; putting $g_{\alpha\beta} = z^{-1}_{\alpha\beta}\Delta_{\alpha\beta}$, we obtain 1-cochains $z_{\alpha\beta}$, $\theta_{\alpha\beta}$, $g_{\alpha\beta}$ taking values in $S^1$, $\mathbb{R}$, $\mathrm{pin}(Q)$, respectively.

It is clear that $g_{\alpha\beta}$ lifts the transition functions $\delta\circ\Delta_{\alpha\beta}$ of $O(M)$. Due to the centrality of $S^1$ in $\Delta(Q)$, $g_{\alpha\beta\gamma}$ defined on the nonempty intersection $U_\alpha \cap U_\beta \cap U_\gamma$ by $g_{\alpha\beta\gamma} = g_{\beta\gamma} g^{-1}_{\alpha\gamma} g_{\alpha\beta}$, determines a Čech 2-cocycle which measures the pinorial obstruction $\mathcal{W}_2$. Consider the commutative diagram of groups with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{C} & \xrightarrow{\ m\ } & \mathbb{C}^* & \longrightarrow & 1 \\
 & & \Big\| & & \Big\uparrow{\scriptstyle \frac{1}{2}\cdot} & & \Big\uparrow{\scriptstyle k} & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow[2\cdot]{} & \mathbb{Z} & \xrightarrow[\bmod 2]{} & \mathbb{Z}_2 \simeq S^0 & \longrightarrow & 1
\end{array}$$

From the resulting cohomology diagram, it follows immediately that $\mathcal{W}_2 = (\mathrm{mod}\ 2)_*(C_1)$.

Conversely, if such a condition is satisfied for any integral class $C_1$, in virtue of commutativity of the corresponding cohomological diagram, we get

$$\begin{aligned}\mathcal{W}_3 &= B^2_m \circ B^1_f \circ \mathrm{ext}_*[O(M)] = B^2_m \circ k_* \circ B^1_\sigma[O(M)] = B^2_m \circ k_*(\mathcal{W}_2) \\ &= B^2_m \circ (k \circ \mathrm{mod}\ 2)_*(C_1) = B^2_m \circ m_* \circ (\tfrac{1}{2}.)_*(C_1) = 0\ ,\end{aligned}$$

due to the exactness.

### 2.4. TOROGONAL AND PINORIAL CONNECTIONS

Due to the torogonal diagram, the torogonal Lie algebra splits as a direct sum of ideals

$$\mathscr{L}(\Delta(Q)) = \mathscr{L}(\mathrm{pin}(Q)) \oplus \mathrm{i}\mathbb{R} = \mathscr{L}(O(Q)) \oplus \mathrm{i}\mathbb{R}$$

and the differential of the spinor norm morphism is the twofold of the projection of $\mathscr{L}(\Delta(Q))$ onto $\mathrm{i}\mathbb{R}$.

**Definitions.** For corresponding structures, a Euclidean (resp. torogonal (resp. pinorial)) connection is a principal connection on the orthonormal $O(M)$ (resp. torogonal $\Delta(M)$ (resp. pinorial $\mathrm{pin}(M)$)) bundle; let $\mathscr{C}(O(M))$ (resp. $\mathscr{C}(\Delta(M))$ (resp. $\mathscr{C}(\mathrm{pin}(M))$)) be the space of all connection 1-forms.

**Theorem** [16b]. *Using the concept of mappings of connections [11], the direct image map $\delta \times N$: $\mathscr{C}(\Delta(M)) \to \mathscr{C}(O(M)) \times \mathscr{C}(N(M))$ (resp. $\sigma$: $\mathscr{C}(\mathrm{pin}(M)) \to \mathscr{C}(O(M))$) is a bijection; any Euclidean connection $\theta \in \mathscr{C}(O(M))$ and scalar connection $\gamma \in \mathscr{C}(N(M))$ considered as a principal connection on the torogonal line bundle, lift to torogonal (resp. pinorial) connection $\tau \in \mathscr{C}(\Delta(M))$ (resp. $\tilde{\theta} \in \mathscr{C}(\mathrm{pin}(M))$) according to the relation*

$$\tau = (\mathrm{T}\sigma)^{-1} \circ \delta^*(\theta) + \tfrac{1}{2} N^*(\gamma) \quad (\text{resp. } \tilde{\theta} = (\mathrm{T}\sigma)^{-1} \circ \sigma^*(\theta)),$$

*with the same formula for the corresponding curvature forms. Recall that the differential $\mathrm{T}\sigma$ of the covering $\sigma$ is a Lie algebra isomorphism and $\delta^*(\theta)$ (resp. $\sigma^*(\theta)$) is the pull-back of $\theta$ by the corresponding bundle map $\delta$ (resp. $\sigma$).*

**Remark.** Such a form as $\frac{1}{2}N^*(\gamma)$ is said to be a pseudo-connection on $\Delta(M)$ [8] and is closely related to the Cliffordian excess [4].

From any pinorial representation $\rho$ of $C(Q')$ to a standard spinor space $S$, whose restriction to $\Delta(Q)$ is irreducible, we can construct the previous spinor bundle $\mathscr{S}$ as the associated complex vector bundle $\Delta(M) \times_\rho S$. Then, a covariant derivation of spinor fields, the so-called S.C. connection [10], is uniquely associated to any torogonal connection. It is possible to construct a standard Hermitian structure on $S$ so that the direct image $\rho(\Delta^U(Q))$ (resp. $\rho(\mathrm{pin}^U(Q))$) is contained in the unitary (resp. unitary special) group of $S$.

**Theorem.** *If the structure group $O(Q)$ of $O(M)$ is reducible to $O^U(Q)$, one obtains a Hermitian structure on the spinor bundle, invariant by the torogonal covariant derivation.*

In the particular case of a spacetime model, the existence of a time-orientation involves the existence of a compatible Hermitian structure on the spinor bundle.

**The torogonal integrality criterion.** *There exists a torogonal structure if and only if there exists a real closed 2-form $\omega$ whose real cohomology class satisfies $\mathscr{W}_2 = m_*([\omega/4\pi])$, identifying De Rham and Čech cohomology groups.*

*Proof.* We shall mainly use the complex line bundle theory [12,15]. If $\mathscr{W}_3=0$, for any $\gamma\in\mathscr{C}(N(M))$, there is an invariant Hermitian structure on $N(M)$. Moreover, a real closed 2-form $\omega$ can be defined on $M$, such that the pull-back of $\mathrm{i}\omega$ by the bundle projection $N(M)\to M$ is the scalar curvature $d\gamma$ of the scalar connection. The first Chern class $C_1$ of the above Hermitian structure is just the De Rham cohomology class $[\omega/2\pi]$.

Conversely, if such a condition is satisfied, $[\omega/2\pi]$ is an integral class; according to the integrality criterion of Kostant–Souriau, there is a Hermitian structure on a complex line bundle such that its first Chern class is just $[\omega/2\pi]$; then, $\mathscr{W}_2=(\mathrm{mod}\ 2)_*(C_1)$.

**Remark.** If the rank of $\omega$ is maximal, the torogonal complex line bundle is analogous to the prequantum bundle introduced by Kostant and Souriau over a symplectic manifold.

**Examples.** Given a pinorial structure $\mathrm{pin}(M)$ and a principal $S^1$ bundle $L^*$, a torogonal structure can be obtained as the extension of the fibred product $\mathrm{pin}(M)\times_M L^*$, with respect to the group morphism $\mathrm{pin}(Q)\times S^1\to\Delta(Q)$ which sends $(g, z)$ to $zg$; likewise, a Euclidean connection and an ordinary connection on $L^*$ determine a torogonal connection. The induced torogonal line bundle is the tensor square of $L^*$. This construction is said to be the amalgamation of a pinorial structure with a Hermitian line bundle.

Clearly, any pinorial structure can be naturally extended to a torogonal structure whose torogonal line bundle is trivial.

## 3. Torogonal lifting of distinguished reduction

We shall consider an even $(2n)$-dimensional manifold $M$ endowed with a metric $Q$ having signature $(r, s)$ so that $r+s=2n$.

### 3.1. DISTINGUISHED REDUCTIONS

The Witt index of $Q$ (resp. $Q'$) is the common dimension $t$ (resp. $n$) of any real (resp. complex) maximal isotropic vector subspace of $\mathbb{E}$ (resp. $\mathbb{E}'$); we have $t=n-\frac{1}{2}|r-s|$.

By means of the sesquilinear prolongation of the metric $Q$, defined by the correspondence $(x_1, x_2)\to Q'(x_1, \bar{x}_2)$, the signature effect is preserved. By complexification of its elements, the real orthogonal group $O(Q)$ which acts on $\mathbb{E}'$ operates on the set of complex maximal isotropic subspaces of $\mathbb{E}'$ with transitive type preserving action.

**Theorem** [16b]. *The restriction of the previous sesquilinear prolongation to any maximal isotropic vector subbundle $P$ in the complexified tangent bundle $TM'$ is zero (resp. positive or negative definite according as $r>s$ or $r<s$) on the vector subbundle $P\cap\bar{P}$ (resp. $P^{\pm}$) whose rank is constant equal to $t$ (resp. $n-t$). Such a Sylvester decomposition $P=(P\cap\bar{P})\oplus P^{\pm}$ associated to $P$ bijectively corresponds to the so-called distinguished reduction $O_P$ of the orthonormal frame bundle $O(M)$ to the subgroup $O(\mathbb{P})$ of $O(Q)$ preserving any fixed model $\mathbb{P}$ of $P$ in $\mathbb{E}'$.*

The neutral $(t=n)$ (resp. definite $(t=0)$) type corresponds to $P=\bar{P}$ (*resp.* $P\cap\bar{P}=\{0\}$). Thus, it is not necessary to complexify a real neutral metric.

Due to its matricial form, the isotropy group $O(\mathbb{P})$ is a semidirect product of one nilpotent subgroup (trivial if $t=0$) by a diagonal one isomorphic to the product $\mathrm{GL}(t,\mathbb{R})\times U(n-t)$ (linear and unitary groups); since $O(\mathbb{P})$ is included in the special orthogonal group $\mathrm{SO}(Q)$, we must suppose that the manifold is orientable.

Clearly, the orthogonal of the real maximal isotropic subspace $\mathbb{D}=(\mathbb{P}\cap\bar{\mathbb{P}})\cap\mathbb{E}$ is the real $(2n-t)$-dimensional subspace $\mathbb{G}=(\mathbb{P}+\bar{\mathbb{P}})\cap\mathbb{E}$ and there exists a complex structure on the quotient space $\mathbb{G}/\mathbb{D}$ admitting the primary decomposition

$$(\mathbb{G}/\mathbb{D})'\simeq(\mathbb{G}'/\mathbb{D}')\simeq(\mathbb{P}/\mathbb{P}\cap\bar{\mathbb{P}})\oplus(\bar{\mathbb{P}}/\mathbb{P}\cap\bar{\mathbb{P}})\ .$$

Since $O(\mathbb{P})$ leaves $\mathbb{P}$ stable as $\mathbb{D}$ and $\mathbb{G}$, restriction to $\mathbb{P}$ (resp. $\mathbb{P}\cap\bar{\mathbb{P}}$ (resp. quotient operation)) yields group morphism ind: $O(\mathbb{P})\to\mathrm{GL}_{\mathbb{C}}(\mathbb{P})$ (resp. the real part $\mathscr{R}$: $O(\mathbb{P})\to\mathrm{GL}_{\mathbb{C}}(\mathbb{P}\cap\bar{\mathbb{P}})$ (resp. the complex part $\mathscr{C}$: $O(\mathbb{P})\to U(\mathbb{G}/\mathbb{D})$)), whose determinant prolongation satisfies: $\det\circ\mathrm{ind}=(\det\circ\mathscr{R}).(\det\circ\mathscr{C})$.

The associated vector bundle $O_P\times_{\mathrm{ind}}\mathbb{P}$ is naturally isomorphic to $P$, while the principal bundle $O_P\times_{\mathrm{ind}}\mathrm{GL}_{\mathbb{C}}(\mathbb{P})$ is isomorphic to the whole frame bundle $L(P)$ of $P$. This also implies the existence of a natural isomorphism between complex line bundles

$$\bigwedge^{n} P\simeq O_P\times_{\det\circ\mathrm{ind}}\mathbb{C}\ ,$$

where the first is the $n$th exterior power of $P$.

The previous Bockstein isomorphism $B^1_m$ sends the isomorphism class of the prolongation $O_P\times_{\det\circ\mathrm{ind}}\mathbb{C}^*$, which is just the determinant bundle $\det L(P)=L(P)\times_{\det}\mathbb{C}^*$ of $P$, to the first complex Chern class $C_1(P)$ in $H^2(M,\mathbb{Z})$.

### 3.2. THE TOROGONAL SCISSION

Taking the following inverse images of the isotropy group $O(\mathbb{P})$ namely, $\mathrm{pin}(\mathbb{P})=\sigma^{-1}(O(\mathbb{P}))$ in the even part of $\mathrm{pin}(Q)$ and $\Delta(\mathbb{P})=\delta^{-1}(O(\mathbb{P}))$, we get a similar induced torogonal diagram so that

$$\Delta(\mathbb{P}) \simeq (\mathrm{pin}(\mathbb{P}) \times S^1)/S^0 .$$

Due to the coincidence on the even part of the Clifford algebra $C(Q')$ of twisted and nontwisted adjoint representation, for any $g \in \mathrm{pin}(\mathbb{P})$ and $x \in \mathbb{E}'$ we have $\sigma(g)(x) = gxg^{-1}$, and since $O(\mathbb{P})$ leaves $\mathbb{P}$ stable, then $\sigma(g)(\mathbb{P}) = \mathbb{P}$. Let $f = y_1 y_2 \ldots y_n$ be a standard isotropic vector [4], where the family $\{y_i\}$ $(1 \leqslant i \leqslant n)$ spans $\mathbb{P}$. Clearly, $gfg^{-1} = (\det \circ \mathrm{ind} \circ \sigma)(g) f$ (1).

Consequently the spinor $gf$ belongs at the same time to the minimal left ideal $C(Q')f$ and the minimal right ideal $fC(Q')$; according to the pure spinor concept [2,3], this intersection is a complex line and the pure spinor $gf$ satisfies $gf = \mu f$ where $\mu \in \mathbb{C}^*$. Applying the anti-automorphism $\beta$, we have $f\beta(g) = \mu f$, whence $f\beta(g)\, g = \mu f g$, and since the pinorial group is included in the kernel of the spinor norm, we obtain $f = \mu f g$. Therefore $gfg^{-1} = \mu^2 f$ (2). Comparing (1) and (2), we get $(\det \circ \mathrm{ind} \circ \sigma)(g) = \mu^2$ (3).

We recall that the double covering of the complex linear group $\mathrm{GL}_{\mathbb{C}}(\mathbb{P})$, the so-called complex metalinear group $\mathrm{ML}_{\mathbb{C}}(\mathbb{P})$, can be identified with the set of all pairs $(u, z)$, where $u \in \mathrm{GL}_{\mathbb{C}}(\mathbb{P})$ and $z \in \mathbb{C}^*$ satisfy $\det u = z^2$; the first projection $p_1(u, z) = u$ is the covering map while the second projection $p_2(u, z) = z$ is a holomorphic character such that $\det \circ p_1 = (p_2)^2$. From (3), we can define a group morphism $\widetilde{\mathrm{ind}}$: $\mathrm{pin}(\mathbb{P}) \to \mathrm{ML}_{\mathbb{C}}(\mathbb{P})$ by $\widetilde{\mathrm{ind}}(g) = ((\mathrm{ind} \circ \sigma)(g), \mu)$ which lifts ind: $O(\mathbb{P}) \to \mathrm{GL}_{\mathbb{C}}(\mathbb{P})$. The character $\chi = p_2 \circ \widetilde{\mathrm{ind}}$: $\mathrm{pin}(\mathbb{P}) \to \mathbb{C}^*$ satisfies the following properties: $\chi(\varepsilon) = \varepsilon$ where $\varepsilon \in S^0 = \{\pm 1\}$; $\chi^2 = \det \circ \mathrm{ind} \circ \sigma$ and $|\chi|^2 = |\det \circ \mathscr{R} \circ \sigma|$. Instead of $\chi$, we consider $\hat{\chi} = \chi / |\chi|$: $\mathrm{pin}(\mathbb{P}) \to S^1$. Since $\hat{\chi}(\varepsilon) = \varepsilon$, the homomorphism $\tilde{\chi}$: $\mathrm{pin}(\mathbb{P}) \to \Delta(\mathbb{P})$ defined by $\tilde{\chi}(g) = [g, \hat{\chi}(g)]$, where the square brackets indicate taking equivalence classes mod $S^0$, factors through the covering $\sigma$: $\mathrm{pin}(\mathbb{P}) \to O(\mathbb{P})$ to yield a homomorphism $\eta$: $O(\mathbb{P}) \to \Delta(\mathbb{P})$, namely $\eta(u) = \tilde{\chi}(g)$ for any $g \in \mathrm{pin}(\mathbb{P})$ such that $\sigma(g) = u$, which is a right inverse to the torogonal projection $\delta$: $\Delta(\mathbb{P}) \to O(\mathbb{P})$. Thus, $\eta$ is nothing but that of the desired torogonal scission.

### 3.3. THE TOROGONAL LIFTING

Applying $\eta$ to the transition functions of the distinguished reduction $O_P$, we obtain the distinguished torogonal structure

$$\Delta_P = O_P \times_\eta \Delta(\mathbb{P}) .$$

By straightforward calculus, we have

$$(N \circ \eta)(u) = \frac{(\det \circ \mathscr{R})(u)}{|(\det \circ \mathscr{R})(u)|} (\det \circ \mathscr{C})(u) .$$

If $Q$ is definite $(t=0)$, there is no real part; but, in contrast, when $Q$ is not definite $(t \neq 0)$, we specialize to a real oriented distinguished reduction where the real linear special group $\mathrm{SL}(t, \mathbb{R})$ replaces $\mathrm{GL}(t, \mathbb{R})$ throughout, so that we always get

$N\circ\eta = \det\circ\mathscr{C}$.

Therefore, applying the spinor norm to the transition functions of the torogonal lifting $\Delta_P$, we obtain its prolongation $\Delta_P \times_N S^1 \simeq O_P \times_{N\circ\eta} S^1 \simeq O_P \times_{\det\circ\mathscr{C}} S^1$. The distinguished torogonal line bundle is nothing more than the Hermitian structure constructed on the determinant bundle $\det L(P)$ with first Chern class $C_1(P)$.

**Theorem.** *Any distinguished reduction induces a torogonal structure.*

**Corollary.** *There exists a distinguished spinorial structure if and only if the cohomology class $\frac{1}{2}C_1(P)$ is integral.*

*Proof.* This results from theorem 1.

This existence condition is also equivalent to the existence of a complex metalinear structure regarded as a lifting $\tilde{L}(P)$ of the frame bundle $L(P)$, whose prolongation by means of the character $p_2$ is just the square root of the determinant bundle; the associated complex line bundle $\tilde{L}(P) \times_{p_2} \mathbb{C}$ is said to be the half-$P$-form bundle or the so-called half-isotropic vector bundle.

## 4. The twisting procedure

Now we assume we are given a real oriented distinguished reduction $O_P$ together with a real closed two-form $\omega$ over $M$ such that the real cohomology class $[\omega/2\pi] - \frac{1}{2}C_1(P)$ is integral. According to the integrality criterion of B. Kostant and J.M. Souriau, we can choose a Hermitian line bundle $K^*$ admitting this cohomology class as first Chern class $C_1(K^*)$.

Recall that the cohomology group $H^1(M, S^1)$ acts simply transitively on the set of inequivalent torogonal structures with the operation of tensor product on the set of spinor bundle.

Applying the group morphism $\tau: \Delta(\mathbb{P}) \times S^1 \to \Delta(\mathbb{P})$ defined by $\tau([g, z], z') = [g, zz']$ for $g \in \mathrm{pin}(\mathbb{P})$ and $z \in S^1$, $z' \in S^1$, to the transition functions of the fibred product $\Delta_P \times_M K^*$, we obtain a new torogonal structure $\Delta_P^\omega$. Applying the character $\nu$: $\Delta(\mathbb{P}) \to S^1$ defined by $\nu[g, z] = z\hat{\chi}(g)$ to the transition functions of $\Delta_P^\omega$, we obtain a Hermitian line bundle $L^* = \Delta_P^\omega \times_\nu S^1$ whose first Chern class $C_1(L^*)$ satisfies

$$C_1(L^*) = C_1(K^*) + C_1(P) = [\omega/2\pi] + \tfrac{1}{2}C_1(P) .$$

But if we apply the other characters at our disposal, namely $\det\circ\mathscr{C}\circ\delta$ and $N$, to the transition functions of $\Delta_P^\omega$ we obtain as Hermitian line bundles, the determinant bundle $\det L(P)$ and the tensor square of the prequantum bundle $L^*_{\mathrm{K-S}}$ of Kostant–Souriau, with first Chern class $C_1(P)$ and $2[\omega/2\pi]$. We can summarize this discussion in the following diagram:

$$
\begin{array}{ccccc}
 & & \Delta_P \times_M K^* & & \det L(P) \\
 & & \downarrow{\scriptstyle \tau} & \nearrow{\scriptstyle \det \circ \mathscr{C} \circ \delta} & \\
L^* & \xleftarrow{\nu} & \Delta_P^{\omega} & & \\
 & & \downarrow{\scriptstyle \delta} & \searrow{\scriptstyle N} & \\
 & & O_P & & (L^*_{\text{K-S}})^2
\end{array}
$$

**Remark.** It is well known that the complex line bundle $L^*$ is the only one that may be a candidate for a quantum bundle in the geometric quantization theory.

## 5. Examples

**Example 1.** Given an orientable, $2n$-dimensional Riemannian manifold, any distinguished reduction is nothing more than a compatible almost complex structure $\mathscr{I}$ ($\mathscr{I}^2 = -\text{id}$), turning $M$ into an almost Hermitian manifold; the correspondence assigns to $\mathscr{I}$ its eigenbundle $P$ for eigenvalue $+\mathrm{i}$. The unitary frame bundle is the reduction of the orthonormal frame bundle $O(M)$ from $\mathrm{SO}(2n)$ to $U(n)$.

**Remark.** Note that $M = S^{2n}$ does not admit any compatible almost complex structure if $n \neq 1, 3$.

**Example 2. The symplectic case.** Let $M$ be a symplectic $2n$-dimensional manifold with a nonsingular closed two-form $\omega$. since $U(n)$ is a maximal compact subgroup of the symplectic group $\mathrm{Sp}(\omega) = \mathrm{Sp}(2n, \mathbb{R})$, the symplectic frame bundle is reducible to unitary frame bundle and there exist compatible almost complex structure $\mathscr{I}$ and subordinate positive definite metric $Q$ such that $Q(x_1, x_2) = \omega(\mathscr{I} x_2, x_1)$, the previous eigenbundle $P$ being maximal isotropic and Lagrangian. The complex part of the group morphism ind lifts to isomorphism between $\mathrm{pin}(\mathbb{P})$ (resp. $\Delta(\mathbb{P})$) and the meta-unitary (resp. toro-unitary) group $\mathrm{MU}(n)$ (resp. $\Delta U(n)$), regarded as double covering (resp. central extension by $S^1$) of the unitary group; the embedding $U(n) \to \mathrm{SO}(2n)$ (resp. $U(n) \to \mathrm{Sp}(2n, \mathbb{R})$) lifts to an embedding $\Delta U(n) \to \Delta(Q)$ (resp. $\Delta U(n) \to \Delta(\omega)$), where $\Delta(\omega)$ is the so-called toroplectic group as a central extension by $S^1$ of $\mathrm{Sp}(\omega)$ or equivalently, $\Delta(\omega) = (\mathrm{Mp}(\omega) \times S^1)/S^0$ where $\mathrm{Mp}(\omega)$ is the metaplectic group as a double covering of $\mathrm{Sp}(\omega)$. Observe that we have the toroplectic (resp. toro-unitary) counterpart of the torogonal diagram [8].

Thus, there exists a subordinate torogonal (resp. a canonical toroplectic) structure, admitting a common toro-unitary structure.

The canonical prolongation to $S^1$ of all these is just the determinant bundle of

$M$ whose first Chern class is nothing else than the first Chern class of the unitary frame bundle; moreover, it is independent of the choice of $\mathscr{I}$ by homotopic deformation. This canonical class $C_1(M, \omega)$ in $H^2(M, \mathbb{Z})$ reduces mod 2 to the second Stiefel–Whitney class $W_2$ in $H^2(M, \mathbb{Z}_2)$. The integrality of $\frac{1}{2}C_1(M, \omega)$ anyway characterizes metaplectic structure, subordinate spinorial structure and half-form bundle.

For a Kählerian structure regarded as symplectic structure whose underlying almost Hermitian structure is Frobenius-integrable, namely the Nijenhuis torsion of $\mathscr{I}$ vanishes, the Levi-Civita connection is bi-Lagrangian and preserves the Hermitian structure [14]. The Kählerian Ricci form $\Omega$ defined by $\Omega(x_1, x_2) = \mathrm{Ric}(x_1, \mathscr{I}x_2)$, where Ric denotes the Ricci tensor, is related to the canonical class by

$$C_1(M, \omega) = [\Omega/2\pi] .$$

**Example 3. The harmonic oscillator.** In the geometric quantization of energy surfaces of the $n$-dimensional harmonic oscillator, we consider the complex projective space $M = P_{n-1}(\mathbb{C})$ as its reduced phase space. Since it is an Einstein Kählerian manifold, we can choose the positive generator $[\omega/2\pi]$ of $H^2(M, \mathbb{Z})$ in such a way that $\mathrm{Ric} = nQ$, whence $\Omega = n\omega$ and then $C_1(M, \omega) = [n\omega/2\pi]$. For odd $n$, $\frac{1}{2}C_1(M, \omega)$ is half integral and the second Stiefel–Whitney class is nonzero; therefore, the conventional Kostant–Souriau quantization procedure breaks down. To apply the previous twisting procedure, we must find, for instance, a symplectic form $e\omega$ such that

$$[e\omega/2\pi] - \tfrac{1}{2}C_1(M, \omega) = [(e - \tfrac{1}{2}n)\omega/2\pi]$$

is integral; the requirement $e \in \mathbb{N} + \frac{1}{2}n$ is nothing but that of the Schrödinger energy levels.

**Example 4. Mechanical systems.** For any given $n$-dimensional configuration space $V$ we consider the state space $M = \mathrm{T}V$ instead of the phase space $\mathrm{T}^*V$, endowed with a kinetic energy function $T$ whose vertical differential $\mathrm{d}_V T$, which is the pull-back of the Liouville form under the associated Legendre transformation, determines an exact symplectic structure $\omega = \mathrm{d}\mathrm{d}_V T$ if the system is regular.

The diagonal lifting [17] of any Riemannian metric $g$ on $V$ with respect to the corresponding Levi-Civita connection is a Riemannian metric $Q$ on $M$ subordinate to an almost Kählerian structure, which is Kählerian if and only if $V$ is locally Euclidean. In such cases, the twisting procedure is possible if and only if $W_2(\mathrm{T}V) = 0$, or according to [16a] if and only if $W_1(V)^2 = 0$, where $W_1(V)$ is the first Stiefel–Whitney class of $V$.

The horizontal lifting [17] of any pseudo-Riemannian metric $g$ on $V$ with respect to any Euclidean connection is a neutral pseudo-Riemannian metric $Q$ so that the double tangent bundle is the Whitney sum of the vertical and horizontal distributions. The twisting procedure is always possible and the resulting quantum bundle is the tensor product of the trivial Hermitian structure $TV \times S^1$ with connection $\mathrm{i}d_V T + \mathrm{d}z/z$ by the half-vertical-form bundle, which is trivial if $V$ is locally pseudo-Euclidean.

For example, considering a free charged relativistic particle moving in exterior gravitation field $(V, g)$, where $V$ is the spacetime, and subjected to electromagnetic field $p_V^*(\mathrm{d}A)$ (resp. $p_V^*(E)$), namely the pull-back of the potential 1-form $A$ on $V$ (resp. real closed 2-form $E$ on $V$) by the bundle projection $p_V$: $TV \to V$, the Lorentz equations can be obtained by means of the symplectic structure $\omega_e = \mathrm{d}d_V T + e p_V^*(\mathrm{d}A)$ (resp. $\omega_e = \mathrm{d}d_V T + e p_V^*(E)$), where $e$ is the charge. The twisting procedure is always (resp. if and only if the real cohomology class $[eE/2\pi]$ is integral in $H^2(V, \mathbb{R})$) possible.

Note that the Robertson–Walker cosmological models certainly satisfy the vanishing condition of the torogonal obstruction.

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